4D-2 Show that the average value of $1/x$ over the interval $[a, 2a]$ is of the form C/a, where C is a constant independent of a. (Assume $a > 0$.)

$$\text{Average} = \frac{\int_a^{2a} \frac{1}{x}}{2a - a}$$

$$= \frac{\ln x \Big|_a^{2a}}{a}$$

$$= \frac{1}{a}(\ln 2a - \ln a)$$

$$= \frac{\ln 2}{a}$$

$$\ln 2a - \ln a$$
$$= \ln 2 + \ln a - \ln a$$
$$= \ln 2$$

4D-3 A point is moving along the x-axis, with distance function given by $x = s(t)$. Show that over a time interval $[a, b]$, the average value of its velocity $v(t)$ is the same as its average velocity over this interval.

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Average velocity $v(t) = \frac{1}{b-a} \int_a^b v(t)\,dt$

$x = s(t) \Rightarrow$ Average velocity over $[a, b]$

$$= \frac{s(b) - s(a)}{b - a}$$

With the First Fundamental Theorem of Calculus,

$F(b) - F(a) = \int_a^b v(t)\,dt$ for some antiderivative F.

Since $\frac{d}{dt} s(t) = v(t)$, then $s(b) - s(a) = \int_a^b v(t)\,dt$.

$\Rightarrow$ Average velocity $v(t) = \frac{1}{b-a} \int_a^b v(t)\,dt$

$$= \frac{1}{b-a}(s(b) - s(a))$$

$$= \frac{s(b) - s(a)}{b - a}$$

$\therefore$ The average value of $v(t)$ over $[a, b]$ is the average velocity over the same interval.

4D-5 If the average value of $f(t)$ between 0 and x is given by the function $g(x)$, express $f(x)$ in terms of $g(x)$.

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$$\text{Average} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\Rightarrow \quad g(x) = \frac{1}{x-0}\int_0^x f(t)\,dt$$

$$= \frac{1}{x}\int_0^x f(t)\,dt$$

$$\Rightarrow \quad \int_0^x f(t)\,dt = x \cdot g(x)$$

$$\Rightarrow \quad f(x) = 1 \cdot g(x) + x \cdot g'(x)$$

$$= g(x) + x \cdot g'(x)$$